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A SPARING MODEL FOR SEA-BASED AIRCRAFT PARTS(U) CENTER  
FOR NAVAL ANALYSES ALEXANDRIA VA NAVAL WARFARE  
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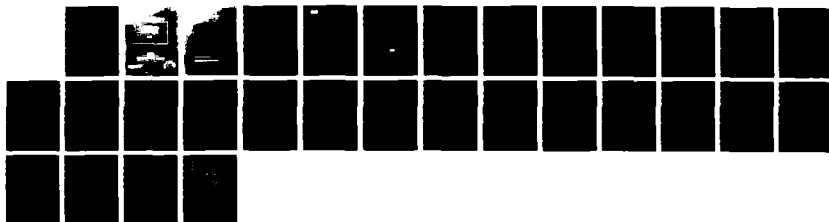
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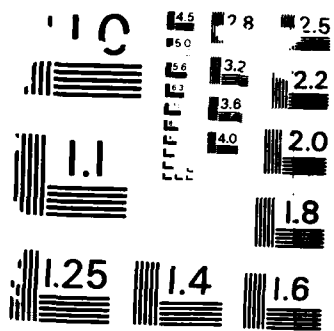
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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
1a. REPORT SECURITY CLASSIFICATION Unclassified			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION / AVAILABILITY OF REPORT APPROVED FOR PUBLIC RELEASE, DISTRIBUTION UNLIMITED		
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) CRM 87-226			5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION Center for Naval Analyses		6b. OFFICE SYMBOL (If applicable) CNA	7a. NAME OF MONITORING ORGANIZATION Office of Chief of Naval Operations (OP-04)		
6c. ADDRESS (City, State, and ZIP Code) 4401 Ford Avenue Alexandria, Virginia 22302-0268			7b. ADDRESS (City, State, and ZIP Code) Navy Department Washington, D.C. 20350-2000		
8a. NAME OF FUNDING / SPONSORING ORGANIZATION Office of Naval Research		8b. OFFICE SYMBOL (If applicable) ONR	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-87-C-0001		
8c. ADDRESS (City, State, and ZIP Code) 800 North Quincy Street Arlington, Virginia 22217			10. SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO. 65154N	PROJECT NO. R0148	TASK NO.
					WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) A Sparing Model for Sea-Based Aircraft Parts					
12. PERSONAL AUTHOR(S) Yair Eitan					
13a. TYPE OF REPORT Final		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Year, Month, Day) November 1987	
				15. PAGE COUNT 28	
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP			
12	04		Aircraft maintenance, Carrier based aircraft, demand, Failure		
15	05		Queuing theory, Sea-based aircraft, Spare parts, Statistical analysis		
19. ABSTRACT (Continue on reverse if necessary and identify by block number)  This research memorandum describes a queueing model used to analyze a sparing decision for a part with general probabilistic demand. It also describes an extension of the model to include discriminating treatment of the repair and resupply pipelines. The final section applies the model to an illustrative example.					
20. DISTRIBUTION / AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION Unclassified		
22a. NAME OF RESPONSIBLE INDIVIDUAL Captain G. Downer			22b. TELEPHONE (Include Area Code)		22c. OFFICE SYMBOL

DD Form 1473, JUN 86

Previous editions are obsolete.

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1 December 1987

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## **A SPARING MODEL FOR SEA-BASED AIRCRAFT PARTS**

Yair Eitan

*Naval Warfare Operations Division*

A Division of



Hudson Institute

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**CENTER FOR NAVAL ANALYSES**

4401 Ford Avenue • Post Office Box 16268 • Alexandria, Virginia 22302-0268

## ABSTRACT

This research memorandum describes a queueing model used to analyze a sparing decision for a part with general probabilistic demand. It also describes an extension of the model to include discriminating treatment of the *repair* and *resupply* pipelines. The final section applies the model to an illustrative example.



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## INTRODUCTION

Occasionally, when a sea-based aircraft is being repaired or inspected, a certain part needs to be replaced. A prevailing assumption [1 and 2] is that the number of such occurrences during a specific time period can be approximated with a Poisson probability distribution. An underlying assumption leading to this probability is that the expected number of failed parts depends only on the number of flight hours in that time period. Recent research at the Center for Naval Analyses (CNA) suggests that this may not be the case [3, 4, and 5].

Another characteristic of the Poisson probability distribution is that there is a positive probability that a very large (theoretically: infinite) number of parts will fail during a large enough time unit. In practice, however, the probability that the number of failed parts will be greater than the total number of parts (used or spared) equals zero.

This paper provides a tool for analyzing a sparing policy for a part for which the demand is not necessarily Poisson and for which there is a limit on the number of parts. This tool can be used as the single-item, single-echelon statistical analysis component of a more complex optimization model. The next section formulates a queueing model with exponential service, ample servers, and finite capacity. The model's general application, extension to hyperexponential service, and application to a numerical example are considered in the subsequent section, and the last section contains some concluding remarks.

## MODEL

When an aircraft is being repaired or inspected and a certain part is found to have failed, it is replaced immediately, if a spare part is available. The availability of spares is determined by (1) the number of spares initially stocked on ship; (2) the number of "new" spares, that is, the number of parts repaired on ship or ordered and received from shore; and (3) the demand for spares already satisfied, that is, the number of parts already used to replace failed parts. Simultaneous with the rise of the demand for spares, the removed part enters a queueing system (figure 1).

In queueing theory terminology [6 and 7], the removed parts are *customers* and they *arrive* to receive *service* provided by *servers*. A five-part descriptor,  $A/B/C/F/K$ , is used to summarize symbolically the information regarding a queueing system:  $A$  is the interarrival-time probability distribution,  $B$  is the service-time probability distribution,  $C$  is the number of servers,  $F$  is the system's capacity, and  $K$  is the size of the customer population.  $A$  and  $B$  take on values from, among others, the following set of symbols whose interpretation is given in terms of distributions within parentheses:  $M$  (exponential),  $E_n$  ( $n$ -stage Erlangian),  $H_N$  ( $N$ -stage hyperexponential),  $D$  (deterministic), and  $G$  (general).  $C$ ,  $F$ , and  $K$  are either integers or infinity, with infinity ( $\infty$ ) symbolizing the fact that whatever the actual number is, it is large enough to yield a certain attribute of the system.

In this notation, the currently used model—the single-item, single-echelon component of the Multi-Item Multi-Echelon (MIME) inventory model—is an  $M/M/\infty/\infty/\infty$  queueing system. That is, the interarrival and service times are exponential; the number of servers is large enough so that no customer has to wait for service to begin; the system is uncapacitated; and the size of the calling population is large enough for the demand rate to remain constant, regardless of the number of customers already in the system. The number of customers, i.e., failed parts, present in this system has a non-zero probability of being larger than the total number of parts. To remedy this situation, one can use the  $M/M/\infty/m/\infty$  queueing system, with  $m$  denoting the total number of parts.

It is of interest to note that the finite-number-of-parts issue was addressed by a finite-system-capacity model rather than by a finite-calling-population-size model. The reason for this action is that the model needed to reflect the fact that the demand rate remains constant, regardless of the number of customers already in the system. Although the main reason for a constant demand rate is that operable aircraft must also complete missions of inoperable aircraft, it is, perhaps, enhanced by cannibalization—an issue not explicitly addressed in this paper.

Although the  $M/M/\infty/m/\infty$  finite-capacity system is an improvement over the  $M/M/\infty/\infty/\infty$  uncapacitated system, it still lacks the generality to accommodate any interarrival-time distribution. Thus, the  $G/M/\infty/m/\infty$  queueing system is considered next. Because no more than  $m$  customers can be present in the system, there is no loss of generality in assuming that only  $n$  servers provide service, as long as  $n$  assumes

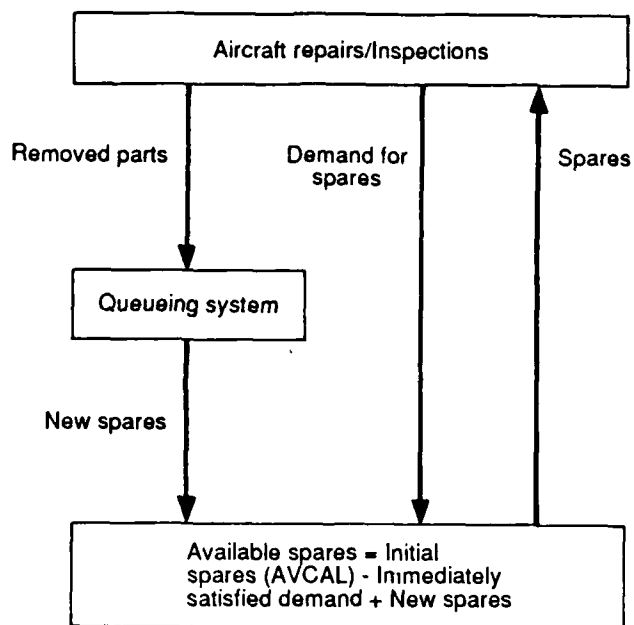


FIG. 1: MODEL SUMMARY

a value from the set  $\{m, m-1, \dots, \infty\}$ . In particular, when  $n = m$ , the result is the  $G/M/m/m/\infty$  queueing system.

This system, along with its special cases, is well solved for the exponential ( $G = M$ ) case 6 and 7. The only explicit attack on the  $G/M/m/m/\infty$  system is by Takács (see the chapter on Telephone Traffic Processes of his book [8]). Although his result is significant in the theoretical sense, its use is limited by the fact that it is given as the Laplace transforms of the probabilities of interest. That is, to obtain the probabilities, the Laplace transforms must be *inversed*. This cannot be done in a mechanized manner. The work described in this paper follows the solution approach to the infinite-capacity,  $n$ -server case 6 and 7. Results are expressed, as a matter of convenience, in terms of Laplace transforms of the interarrival time density, but they do not require *inversion*. The appendix shows that the solution here reduces in the exponential case to the well-known results. The remainder of this section is devoted to a solution procedure for the  $G/M/m/m/\infty$  queueing system.

Because the exponential service time is memoryless, at each arrival instance the system starts from the beginning, in the sense of statistical behavior. Thus, the arrival instances constitute the regeneration points of an *imbedded* Markov chain. The states of this chain are the number of customers found in the system, and the probability of transition from state  $i$  to state  $j$  is the conditional probability of the  $n-1^{st}$  arrival to find  $j$  customers in the system, given that the  $n^{th}$  arrival found  $i$  customers in the system. If  $q_n$  is the number of customers found in the system by the  $n^{th}$  arrival, then  $\{q_n\}$  is a discrete-state Markov chain and its transition probability  $p_{ij}$  is defined by equation 1 as

$$p_{ij} \equiv \Pr\{q_{n-1} = j \mid q_n = i\} \quad (1)$$

This transition probability is nothing but:

$$\begin{aligned} p_{ij} &= \Pr\{(i' - j) \text{ customers are served during an interarrival time}\} \\ &= \int_0^\infty \Pr\{(i' - j) \text{ customers are served during an interarrival} \\ &\quad \text{time of length } t\} dA(t) \end{aligned} \quad (2)$$

which equals:

$$p_{ij} = \begin{cases} \int_0^\infty \binom{i'}{j} (1 - e^{-\mu t})^{i'-j} e^{-\mu t} dA(t); & i = 0, 1, \dots, m; \quad j = 0, 1, \dots, i' \\ 0; & \text{otherwise} \end{cases} \quad (3)$$

where  $i' = \min\{i-1, m\}$  and the integration is Riemann-Stieltjes. It should be noted that  $p_{mj} = p_{m-1,j}$  for all  $j$ .

Figure 2 presents a diagram with the transition probabilities into and out of state  $i$ .

Replacing  $(1 - e^{-\mu t})^{i'-j}$  with its binomial expansion  $\sum_{k=0}^{i'-j} \binom{i'-j}{k} (-1)^k (e^{-\mu t})^k$ , and noting that  $\int_0^\infty (e^{-\mu t})^{k-j} dA(t)$  is the expected value of  $e^{-\mu t(k-j)}$  with respect to random variable  $t$  (that is, the Laplace transform of  $dA(t)$  evaluated at  $\mu(k-j)$ ,  $A^*(\mu(k-j))$ ), equation 3 can be rewritten in the following, computationally tractable, form:

$$p_{ij} = \begin{cases} \binom{i'}{j} \sum_{k=0}^{i'-j} \binom{i'-j}{k} (-1)^k A^*(k-j)\mu & i = 0, 1, \dots, m; j = 0, 1, \dots, i' \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where, as before,  $i' = \min\{i+1, m\}$ .

Let  $r_k = \lim_{n \rightarrow \infty} Pr\{q_n = k\}$  denote the (steady-state, unconditional) probability of a random arrival to find  $k$  customers in the system. It should be noted that finite system capacity ensures that steady-state equilibrium is attainable. Having found the (one step, conditional) transition probabilities given by equation 4,  $r_k$  can be found by solving the following  $(m+1)$  by  $(m+1)$  linear-equations system:

$$r_k = \sum_{n=k+1}^m r_n p_{nk} \quad k = 1, 2, \dots, m \quad (5)$$

$$\sum_{n=0}^m r_n = 1 \quad (6)$$

Toward this end, equation 5 can be rewritten as,

$$r_{k+1} = \left[ r_k + \sum_{n=k}^m r_n p_{nk} \right] p_{k+1,k} \quad k = 1, 2, \dots, m-1 \quad (7)$$

If, based on observations, theoretical knowledge or computational instability  $r_m$  equals 0 (or is small enough),  $m$  can be replaced by  $m-1$ ; thus, there is no loss of generality in assuming  $r_m = 0$ , and the following ratio can be defined:

$$c_k = r_{m-k} / r_{m-1}, k = 0, 1, \dots, m \quad (8)$$

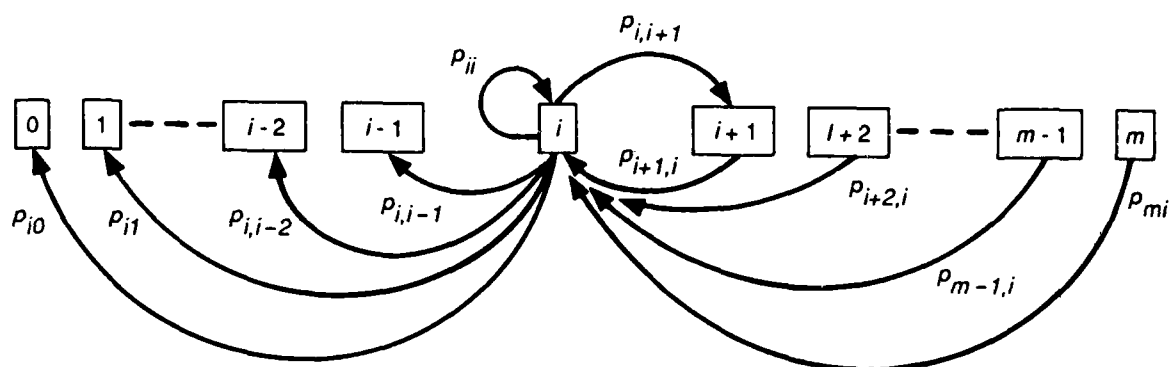


FIG. 2: TRANSITION PROBABILITIES

As implied by equation 6,  $r_m$  must equal  $1 - \sum_{k=0}^m c_k$ ; therefore equation 8 can be rewritten as

$$r_k = c_{m-k} - \sum_{n=0}^m c_n : k = 0, 1, \dots, m-1 \quad (9)$$

That is, given the transition probabilities  $p_{ij}$  calculated in the previous step, the system of linear equations can be solved by finding  $c_k$ :  $k = 0, 1, \dots, m$ . This is done by the following recursive formulae:

$$\begin{aligned} c_0 &\equiv 1 \\ c_{k+1} &= \left[ c_k - \sum_{n=0}^k c_n p_{m-n, m-k} \right] p_{m-(k+1), m-k} : \\ &\quad k = 0, 1, \dots, m-1 \end{aligned} \quad (10)$$



## APPLICATION

An application of the model developed in the previous section must address the following issues:

- What are the queueing system's characteristics, or what can be said about service?
- What is the input to the queueing system, or what can be said about arrivals?
- What is the significance of the queueing system's output to decision makers?

## SERVICE

Because the model presented here uses queueing theory, the information about the number of customers served during a specific time period is conveyed as information about the time it takes to serve a customer. Because a customer is a removed part and service includes off-ship resupply (when on-ship repair is not possible), it is natural to consider service time in calendar time rather than, for example, flight hours or number of flights. Average service time equals average on-ship repair time weighted by the probability of repair aboard ship, plus the average off-ship resupply time weighted by the probability of off-ship resupply. For example, in the case of a pulse decoder, 5 percent of the removed parts can be repaired aboard ship, but the remaining 95 percent must be sent off ship. The average on-ship repair time is 4 days, while the average off-ship resupply time is 44 days. Thus, the average service time equals 42 days ( $.05 \times 4 + .95 \times 44 = 42$ ). This holds true for any probability distribution of service time. To facilitate numerical comparison with the currently used model and for the sake of ease of exposition, service time is assumed to have an exponential distribution (with parameter  $\mu$  that equals the reciprocal of the average service time). Such a combination of on-ship repair and off-ship resupply implicitly assumes that a part repaired on ship is identical to a shore-supplied part.

## ARRIVALS

The information pertaining to the number of arrivals is conveyed again as information about interarrival times. The choice of interarrival times to be in calendar time is not a natural one, because it necessitates "stretching" flights into calendar time; however, that choice is dictated by the need to be compatible with service times and to allow consideration of factors contributing to the rise of demand for spare parts—that is, arrivals—that do not depend only on flight hours. Two approaches to determining the probability distribution of interarrival times are possible: parametric and nonparametric. In the parametric case, a probability distribution is determined by theoretical knowledge of the underlying process or by closeness of shape to the shape of real-life

data: the parameters are estimated using real-life data. In this case, the Laplace transform of the appropriate (theoretical) probability density function, with the appropriate parameters, is used in equation 4. In the nonparametric case, an explicit consideration of a probability distribution and its parameters is by-passed, and the data are used directly to calculate the Laplace transform in the following way. Assume that  $n$  observations of interarrival (calendar) times are available, from either real life or a simulation. Denote these observations by  $t_1, t_2, \dots, t_n$ . The Laplace transform in this case is:

$$A[(k-j)\mu] = \sum_{i=1}^n e^{-(k-j)\mu t_i} \frac{1}{n} \quad (11)$$

If  $n$  is too large a number, observations can be grouped together, creating an empirical distribution, and in equation 11 replacing  $t_i$  by the middle of the group's range,  $n$  by the number of groups, and  $\frac{1}{n}$  by the proportion of observations falling in that group. (It should be noted that information can be lost while grouping.)

## OUTPUT

Probabilistic knowledge of the number of parts found in the queueing system allows for probabilistic knowledge of the number of parts available for operational use. In particular, the  $r_k$ s can be used to calculate, for a given outfitting level  $s$ , the expected number of back orders found at arrival points  $B(s)$ , and the probability that demand can be fulfilled immediately upon arrival, i.e., the fill rate  $F(s)$ :

$$B(s) = \sum_{k=s-1}^m (k-s)r_k \quad (12)$$

$$F(s) = \sum_{k=0}^{s-1} r_k : s \geq 1 \quad , \quad F(0) \equiv 0 \quad (13)$$

Before proceeding to the numerical example, let us revisit the assumption that service time—which includes on-ship repair and off-ship resupply—has an exponential distribution. For the currently used model, this issue is purely theoretical: its results hold regardless of the actual distribution of service time. For the model proposed here, the matter is crucial: only because of the memorylessness of the exponential distribution were we able to find regeneration points, an embedded Markov chain, and to complete the analysis. In a queueing system with two types of servers, each type having an exponential service time (with different parameters), the combined service time is exponential only when all servers are busy. In this case, there are two Poisson outputs of completed services, whose combination is a Poisson output. In this model, in general, not all servers

are busy. Assuming that each type of service (on-ship and off-ship) can be approximated as having an exponential distribution, the combined service can be viewed as having a two-stage hyperexponential distribution. This approach is similar to the one taken in 9.

Let  $p'$  be the probability of repair aboard ship.  $p'$  is estimated by  $p$ , which equals the number of parts removed that can be repaired aboard ship, divided by the sum of the number of removed parts that can be repaired aboard ship and the number of removed parts that cannot be repaired aboard ship. Let  $\mu_1$  denote the reciprocal of the average on-ship repair time, and  $\mu_2$  denote the reciprocal of the average off-ship resupply time. The density function of the combined service time is given by:

$$h_2(t) = p\mu_1 e^{-\mu_1 t} + (1 - p)\mu_2 e^{-\mu_2 t} \quad (14)$$

Equation 14 can be interpreted as a conditional density. Thus, the resulting  $G/H_2 \propto m \propto$  queueing system can be analyzed in the following way:

1. Replace  $\mu$  of equation with  $\mu_1$ .
2. In equations 5 through 10, replace  $r_k$  with  $r_k^1$ .
3. Repeat steps 1 and 2, this time with  $\mu_2$  and  $r_k^2$ .
4. Let  $r_k = pr_k^1 + (1 - p)r_k^2$ ;  $k = 0, 1, \dots, m$ .

To illustrate the subject, consider the following numerical example. There are four aircraft, each using one unit of the part of interest. Each aircraft flies once a day for four hours. The observed average failure rate is two units per 100 flight hours. With an average flight length of four hours, this translates into eight failures per 100 failures are 6 to 3 to 1, respectively, for equal times. The proportions of time spent in each environment are 0.5 24, 19.5 24, and 4 24, respectively. The observed average service time is five days for on-ship repair, which happens 20 percent of the time. In the remaining 80 percent of the time when the repair cannot be done on ship, the average resupply time is 20 days.

Four ways to model demand for spares are considered here:

- An infinite number of parts, exponential interarrival times, denoted INEX. This is the currently used model and is considered in order to facilitate comparison.
- A finite number of parts,  $(4 \times 1 - 2 = 6)$ , exponential interarrival times, denoted FINE.
- Demand derived from four flights per day, 8/100 probability of failure per flight, denoted LEVY 3.

- A finite number of parts and hyperexponential interarrival times (a mixture of three exponentials, each representing one environment of failure), denoted HYPE.

Two ways to model service times are considered:

- Exponential with parameter  $(0.2 \cdot 5 + 0.8 \cdot 20)^{-1} = 17^{-1}$ , denoted by EXPO.
- Hyperexponential (a mixture of two exponentials, one with parameter  $5^{-1}$ , the other with  $20^{-1}$ ), denoted HYEX.

The results for all eight cases are summarized in terms of expected back orders and fill rates in table 1. It should be noted that in spite of mean rates being equal, different distributions (for demand or service) yield different results. (For INEX, the service distribution does not affect the expected back orders.) The infinite-number-of-parts case yields the largest number of back orders and the smallest fill rates. This is not surprising, since "parts" keep failing after all real parts have failed.

The closeness between FINE and LEVY can be explained by recalling the Poisson approximation to the Binomial. The better results produced by HYPE are somewhat surprising, and they hint to the value of having a better understanding of the failure process.

TABLE 1  
NUMERICAL RESULTS

<u>Service</u>	<u>Demand</u>			
	<u>INEX</u>	<u>FINE</u>	<u>LEVY</u>	<u>HYPE</u>
Expected back orders				
EXPO	3.472	2.265	2.263	1.986
HYEX	3.472	2.108	2.104	1.878
Fill rate				
EXPO	.0279	.0403	.0384	.0788
HYEX	.1148	.1233	.1241	.1549

## CONCLUSION

In summary, the model presented in this paper can analyze the consequence of an initial outfitting level for a part with general probabilistic demand, a two-step hyperexponential service, and a limit on the number of parts. The two-stage service is readily extendable to any integer-stage hyperexponential service.

In terms of continuing research, a critical area is the programming of the method presented in this paper and its incorporation into a multi-item multi-echelon inventory model. These areas will be reported on in a future research memorandum.

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**APPENDIX**  
**THE EXPONENTIAL CASE**

## APPENDIX

### THE EXPONENTIAL CASE

When the distribution of the interarrival times is exponential with parameter  $\lambda$ ,  $A(t)$  equals  $1 - e^{-\lambda t}$  for  $t \geq 0$  and vanishes otherwise; the Laplace transform,  $A^*$ , evaluated at  $(k-j)\mu$  is given by:

$$A^*[(k-j)\mu] = \frac{\lambda}{\lambda - (k-j)\mu} \quad (\text{A-1})$$

Dividing the numerator and denominator of the right hand side (RHS) of equation A-1 by  $\mu$  and denoting  $\lambda/\mu$  by  $\rho$  yields:

$$A^*[(k-j)\mu] = \frac{\rho}{\rho - k + j} = \frac{\rho}{\rho - j} \frac{1}{\frac{k}{\rho - j} - 1} \quad (\text{A-2})$$

Substituting the RHS of equation A-2 for  $A^*[(k-j)\mu]$  in equation 4 of the main text yields the following transition probabilities for the exponential case:

$$p_{ij} = \binom{i'}{j} \sum_{k=j}^{i'-j} \binom{i' - j}{k} (-1)^k \frac{\rho}{\rho - j} \frac{1}{\frac{k}{\rho - j} - 1} \quad (\text{A-3})$$

Equation A-3 (and its successors in the sequel) is valid for  $i = 0, 1, \dots, m$ ;  $j = 0, 1, \dots, i'$ , where  $i' = \min\{i + 1, m\}$ , and vanishes otherwise.

Now, let us concentrate on the rightmost part of equation A-3.

$$\frac{1}{\frac{k}{\rho - j} - 1} = \frac{1}{\frac{i'-j}{\rho - j} - 1} \frac{1 - 0}{1 - 0} = \int_0^1 x^{\frac{i'-j}{\rho - j}} dx = \int_0^1 \left(x^{\frac{i'-j}{\rho - j}}\right)^k dx \quad (\text{A-4})$$



Replacing the rightmost component of equation A-3 with the RHS of equation A-4 and moving  $\frac{\rho}{\rho-j}$  to the left of the summation yields:

$$\begin{aligned} p_{ij} &= \binom{i'}{j} \frac{\rho}{\rho-j} \sum_{k=0}^{i'-j} \binom{i'-j}{k} (-1)^k \int_0^1 \left(x^{\frac{1}{\rho-j}}\right)^k dx \quad (\text{A-5}) \\ &= \binom{i'}{j} \frac{\rho}{\rho-j} \int_0^1 \left[ \sum_{k=0}^{i'-j} \binom{i'-j}{k} (-1)^k \left(x^{\frac{1}{\rho-j}}\right)^k \right] dx \end{aligned}$$

The bracketed expression is the binomial expansion of  $\left(x^{\frac{1}{\rho-j}} - 1\right)^{i'-j}$ , so equation A-5 reduces to:

$$p_{ij} = \binom{i'}{j} \frac{\rho}{\rho-j} \int_0^1 \left(x^{\frac{1}{\rho-j}} - 1\right)^{i'-j} dx \quad (\text{A-6})$$

Upon substitution of  $(1-y)^{\rho-j}$  for  $x$ , and  $(\rho-j)(1-y)^{\rho-j-1}dy$  for  $dx$ , equation A-6 yields:

$$p_{ij} = \binom{i'}{j} \frac{\rho}{\rho-j} \int_{-1}^0 y^{i'-j} (\rho-j)(1-y)^{\rho-j-1} dy \quad (\text{A-7})$$

Replacing  $y$  with  $-(1-z)$  and  $dy$  with  $dz$ , noting that  $(\rho-j) \frac{\rho}{\rho-j}$  equals  $\rho$  and that  $i'-j$  equals  $(i'-j-1)-1$ , equation A-7 can be rewritten as:

$$\begin{aligned} p_{ij} &= \binom{i'}{j} \rho \int_{-1}^0 (1-z)^{i'-j-1-1} z^{i'-j-1-1} dz \\ &= \binom{i'}{j} \rho B(i'-j-1, \rho-j) \\ &= \binom{i'}{j} \rho \frac{\Gamma(i'-j-1)\Gamma(\rho-j)}{\Gamma(i'-j-1) + (\rho-j)} \\ &= \frac{(i')!}{(i'-j)!j!} \rho \frac{(i'-j)!\Gamma(\rho-j)}{\Gamma(i'-1-\rho)} \end{aligned}$$

or:

$$p_{ij} = \frac{(i')^j}{j!} \rho \frac{\Gamma(\rho - j)}{\Gamma(\rho - i' - 1)} \quad (\text{A-8})$$

$B(\cdot)$  and  $\Gamma(\cdot)$  are the standard Beta and Gamma functions

Substituting the RHS of equation A-8 for  $p_{ij}$  in equation 10 of the main text yields the following result for the exponential case:

$$C_0 = 1. \quad (\text{A-9})$$

$$C_{k-1} = \frac{C_k - \sum_{n=0}^k C_n \frac{(m-n)^{k-1}}{(m-k)^{k-1}} \rho \frac{\Gamma(\rho - m - k)}{\Gamma(\rho - m - n)^{k-1}}}{\frac{(m-k)^{k-1}}{(m-k)^{k-1}} \rho \frac{\Gamma(\rho - m - k)}{\Gamma(\rho - m - k - 1)}}; k = 0, 1, \dots, m-1$$

where  $(m-n)' = \min\{m-n-1, m\}$ .

Using the fact  $\Gamma(\rho - m - k - 1) = (\rho - m - k)\Gamma(\rho - m - k)$  and cancelling out  $\frac{\Gamma(\rho - m - k)}{\Gamma(\rho - m - k)}$  and  $\frac{\Gamma(\rho - m - k)}{\Gamma(\rho - m - k)}$ , equation A-9 may be rewritten as:

$$C_0 = 1. \quad (\text{A-10})$$

$$C_{k-1} = \frac{C_k - \sum_{n=0}^k C_n \frac{(m-n)^{k-1}}{(m-k)^{k-1}} \rho \frac{\Gamma(\rho - m - k)}{\Gamma(\rho - m - n)^{k-1}}}{\frac{k}{\rho - m - k}}; k = 0, 1, \dots, m-1$$

For  $k = 0$ , equation A-10 yields:

$$C_0 = 1. \quad (\text{A-11})$$

$$C_1 = \frac{C_0 - C_0 \frac{m}{m} \rho \frac{\Gamma(\rho - m)}{\Gamma(\rho - m - 1)}}{\frac{1}{\rho - m - 1}} = \left( \frac{\rho - m - 0}{\rho} - 1 \right) C_0 = \frac{m - 0}{\rho} C_0$$

Combining equations A-10 and A-11 yields:

$$\begin{aligned}
 C_0 &= 1, \\
 C_1 &= \frac{m-0}{\rho} C_0, \\
 C_{k+1} &= \frac{C_k - \sum_{n=0}^k C_n \frac{\Gamma(\rho-n)!}{\Gamma(\rho-n-k)!} \rho \frac{\Gamma(\rho-m-k)}{\Gamma(\rho-m-n-k)!}}{\frac{k}{\rho-m-k}}, \\
 k &= 1, 2, \dots, m-1.
 \end{aligned} \tag{A-12}$$

The summation of expressions involving  $\Gamma(\rho - (m - n))! - 1$  is analytically difficult, therefore, the following approach is taken:

$$m - (k - 1)! = (m - k) - 1! = (m - k - 1)(m - k)!$$

Therefore:

$$(m - k)! = \frac{m - (k - 1)!}{m - k - 1} \tag{A-13}$$

Similarly

$$\begin{aligned}
 \Gamma(\rho - m - (k - 1)) &= \Gamma(\rho - m - k) - 1 \\
 &= (\rho - m - k)\Gamma(\rho - m - k)
 \end{aligned}$$

Therefore:

$$\Gamma(\rho - m - k) = \frac{\Gamma(\rho - m - (k - 1))}{\rho - m - k} \tag{A-14}$$

Using equations A-13 and A-14, the summation component of equation A-12 can be rewritten as:

$$\sum_{n=0}^k C_n \frac{(m-n)!}{(m-k)!} \rho \frac{\Gamma(\rho-m-k)}{\Gamma(m-n)'-1-\rho} = \quad (\text{A-15})$$

$$\frac{m-k-1}{\rho-m-k} \sum_{n=0}^k C_n \frac{(m-n)!}{(m-k)!} \rho \frac{\Gamma(\rho-m-(k-1))}{\Gamma(m-n)'-1-\rho}$$

Scrutinizing the RHS of equation A-15 with the help of equation A-8 yields:

$$\sum_{n=0}^k C_n \frac{(m-n)!}{(m-k)!} \rho \frac{\Gamma(\rho-m-k)}{\Gamma(m-n)'-1-\rho} = \frac{m-k-1}{\rho-m-k} \sum_{n=0}^k C_n p_{m-n, m-(k-1)} \quad (\text{A-16})$$

In order to transform the summation component in the RHS of equation A-16 into a more useful form, let us rewrite equation 10 of the main text as:

$$C_0 = 1, \quad (\text{A-17})$$

$$C_k = \frac{C_{k-1} - \sum_{n=0}^{k-1} C_n p_{m-n, m-(k-1)}}{p_{m-k, m-(k-1)}}, \quad k = 1, 2, \dots, m$$

Multiplying both sides by  $p_{m-k, m-(k-1)}$ , adding  $n = k$  to the summation range, and moving the summation to the other side, equation A-17 yields:

$$\sum_{n=0}^k C_n p_{m-n, m-(k-1)} = C_{k-1}, \quad k = 1, 2, \dots, m \quad (\text{A-18})$$

Utilizing equations A-16 and A-18, equation A-12 can be rewritten as:

$$C_0 = 1, \quad (\text{A-19})$$

$$C_1 = \frac{m-0}{\rho} C_0,$$

$$C_{k+1} = \frac{C_k - \frac{n-k-1}{\rho-m-k} C_{k-1}}{\frac{\rho}{\rho-m-k}}, \quad k = 1, 2, \dots, m-1$$

If  $m = 0$ , the system is of no interest at all. If  $m = 1$ , the system is completely described by  $C_0$  and  $C_1$ . If  $m \geq 2$ , equation A-19 with  $k = 1$  reduces to:

$$\begin{aligned} C_0 &= 1 \\ C_1 &= \frac{m-0}{\rho} C_0 \\ C_2 &= \frac{C_1 - \frac{m-1-1}{\rho-m-1} C_0}{\frac{\rho}{\rho-m-1}} = \frac{\rho-m-1}{\rho} C_1 - \frac{m-0}{\rho} C_0 \\ &= \frac{\rho-m-1}{\rho} C_1 - C_1 = \frac{m-1}{\rho} C_1 \end{aligned}$$

Again, if  $m = 2$ , the system is completely described by  $C_0$ ,  $C_1$ , and  $C_2$ . If  $m \geq 3$ , equation A-19 with  $k = 2$  reduces to:

$$\begin{aligned} C_0 &= 1 \\ C_1 &= \frac{m-0}{\rho} C_0 \\ C_2 &= \frac{m-1}{\rho} C_1 \\ C_3 &= \frac{C_2 - \frac{m-2-1}{\rho-m-2} C_1}{\frac{\rho}{\rho-m-2}} \\ &= \frac{\rho-m-2}{\rho} C_2 - \frac{m-1}{\rho} C_1 \\ &= \frac{\rho-m-2}{\rho} C_2 - C_2 = \frac{m-2}{\rho} C_2 \end{aligned}$$

By now, the following recursive relationship is suspected:

$$\begin{aligned} C_0 &= 1 \\ C_{k-1} &= \frac{m-k}{\rho} C_k; \quad k = 0, 1, \dots, m-1 \end{aligned} \tag{A-20}$$

Assume that equation A-20 holds for  $k = 0, 1, \dots, k^* - 1$  for an arbitrary  $k^*$ . To complete the induction process, it must be shown that equation A-20 holds for  $k = k^*$ . If  $k^* = m$ , the process is complete. If  $k^* \leq m - 1$ , equation A-19 with  $k = k^*$  yields:

$$\begin{aligned} C_{k^*-1} &= \frac{C_{k^*} - \frac{m-k^*+1}{\rho-m-k^*} C_{k^*-1}}{\frac{\rho}{\rho-m-k^*}} \\ &= \frac{\rho-m-k^*}{\rho} C_{k^*} - \frac{m-(k^*-1)}{\rho} C_{k^*-1} \end{aligned} \quad (\text{A-21})$$

Because it is assumed that equation A-20 holds for  $k = k^* - 1$ , that is,  $C_{k^*} = \frac{m-(k^*-1)}{\rho} C_{k^*-1}$ , equation A-21 can be rewritten as:

$$C_{k^*-1} = \frac{\rho-m-k^*}{\rho} C_{k^*} - C_{k^*} = \frac{m-k^*}{\rho} C_{k^*} \quad .$$

which completes the induction process.

Having established the validity of the relationship given by equation A-20 for all  $k$ s in the appropriate range, it should be noted that an equivalent form is:

$$C_0 = 1 \quad (\text{A-22})$$

$$C_{k-1} = \prod_{n=0}^{k-1} \frac{m-n}{\rho} ; k = 0, 1, \dots, m-1 \quad .$$

or:

$$C_0 = 1 \quad (\text{A-23})$$

$$C_k = \frac{m!}{(m-k)! \rho^k} ; k = 1, 2, \dots, m \quad .$$

Substituting the RHS of equation A-23 for the  $C$ s in equation 9 of the main text yields the following steady-state probabilities for the exponential case:

$$r_k = \frac{\frac{m!}{m!(m-k)!} \rho^{m-k}}{\sum_{n=0}^m \frac{m!}{m!(m-n)!} \rho^n} = \frac{\frac{m!}{k!} \rho^{m-k}}{\sum_{n=0}^m \frac{m!}{(m-n)!} \rho^{m-n}}; \quad (A-24)$$

$$k = 0, 1, \dots, m.$$

Cancelling out  $\frac{m!}{m!}$  and  $\frac{\rho^m}{\rho^m}$ , looking at  $(m-n)$  as one index, and reversing its order of summation yields:

$$r_k = \frac{\rho^k k!}{\sum_{n=0}^m \rho^n n!}; \quad k = 0, 1, \dots, m. \quad (A-25)$$

which are the steady-state probabilities for an  $M/M/m/m/\infty$  queueing system.

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